

ESTIMATION OF A DISTRIBUTED LAG MODEL

by

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1. **Introduction.** Distributed lag models have, in the last few years, been playing an important role in research in supply and demand analysis, investment behavior and similar problems in econometrics. Koyck (1954), Nerlove (1958) and Klein (1958) are among several who have proposed a number of models for a number of different situations. It was, however, perhaps Fisher (1925) who first introduced the concept of distributed lags. In general, distributed lag models are those in which the dependent variable is expressed as a function of lagged values of the independent variable or the dependent variable or both.

In this paper we shall be concerned with the estimation of only one of the many forms of distributed lags. The particular model considered is derived from an expectation model.

2. **Derivation of the Distributed Lag Model.** We start with the assumption that the supply of a community, Y_t , at a given time t , is a linear function of its expected price, X_t^* , i.e.,

$$Y_t = \beta_0 + \beta_1 X_t^* + \epsilon_t, \quad (1)$$

where

$$X_t^* - X_{t-1}^* = \gamma (X_t - X_{t-1}^*), \quad 0 < \gamma < 1. \quad (2)$$

Equation (2) is one of the possible relationships between the expected price at time t and the actual price at time t , X_t , and

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the expected price at time $t - 1$, X_{t-1}^* . This relationship suggests that the difference in the expected prices at time t and time $t - 1$ is a fraction of the difference between the actual price at time t and the expected price at time $t - 1$. Simplifying equation (2), we have,

$$X_t^* = \gamma X_t + (1 - \gamma) X_{t-1}^* \quad (3)$$

$$\begin{array}{ccc} X_{t-1}^* & = & \gamma X_{t-1} + (1 - \gamma) X_{t-2}^* \\ \vdots & & \vdots \\ & \text{etc.} & \vdots \end{array}$$

Therefore,

$$X_t^* = \gamma X_t + \gamma (1 - \gamma) X_{t-1} + (1 - \gamma)^2 X_{t-2}^*$$

or

$$X_t^* = \sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{t-i} \quad (4)$$

Substitution of equation (4) to equation (1) yields

$$Y_t = \beta_0 + \beta_1 \left[\sum_{i=0}^{\infty} \gamma (1 - \gamma)^i \right] + \epsilon_t, \quad (5)$$

the distributed lag model. The parameters to be estimated in this model are β_0 , β_1 , and γ .

3. Estimation. Procedures. If the ϵ_t are assumed to be normally and independently distributed with mean zero and common variance, the least squares equations for estimating the parameters of interest can be obtained by minimizing

$$L = \sum_1^n \epsilon_t^2 = \sum_1^n (Y_t - \beta_0 - \beta_1 \sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{t-i})^2.$$

Before we minimize equations (6) let us first simplify it to avoid the infinite sum. By equation (5) we have

$$Y_0 = \beta_0 + \beta_1 \left[\sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{-1} \right]$$

then

$$\sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{-1} = \frac{Y_0 - \beta_0}{\beta_1}$$

and since the quantity $\sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{t-i}$ can be written as

$$\sum_{i=0}^{\infty} \gamma (1 - \gamma)^i X_{t-1} = \sum_{i=0}^{t-1} \gamma (1 - \gamma)^i X_{t-i} + (1 - \gamma)^t$$

$$\sum_{j=0}^{\infty} \gamma (1 - \gamma)^j X_{-j}$$

then

$$Y_t = \beta_0 + \beta_1 \left[\sum_{i=0}^{t-1} \gamma (1 - \gamma)^i X_{t-i} \right] + \beta_1 (1 - \gamma)^t$$

$$\sum_{j=0}^{\infty} \gamma (1 - \gamma)^j X_{-j} + \epsilon_t$$

or

$$Y_t = \beta_0 + \beta_1 \left[\sum_{i=0}^{t-1} \gamma (1 - \gamma)^i X_{t-i} \right] + (1 - \gamma)^t (Y_0 - \beta_0) + \epsilon_t.$$

The quantity now to be minimized is

$$L' = \sum_1^n \epsilon_t^2 = \sum_1^n \left[Y_t - \beta_0 - \beta_1 \left[\sum_{i=0}^{t-1} \gamma (1 - \gamma)^i X_{t-i} \right] - (1 - \gamma)^t (Y_0 - \beta_0) \right]^2.$$

Taking the partial derivatives of L' with respect to β_0 , Y_0 , β_1 , and γ , respectively, we obtain the normal equations as

$$\sum_1^n [\epsilon_t] [-1 + (1-\gamma)^t] = 0$$

$$\sum_1^n [\epsilon_t] [-(1-\gamma)^t] = 0$$

$$\sum_1^n [\epsilon_t] \left[-\sum_{i=0}^{t-1} \gamma (1-\gamma)^i X_{t-1} \right] = 0$$

$$\sum_1^n [\epsilon_t] \left[-\beta_1 \left(\sum_{i=0}^{t-1} \gamma (1-\gamma)^i X_{t-1} \right) - \beta_1 \gamma \left(\sum_{i=0}^{t-1} i (1-\gamma)^{i-1} X_{t-1} \right) + t(1-\gamma)^{t-1} (Y_0 - \beta_0) \right] = 0,$$

where

$$\epsilon_t = Y_t - \beta_0 - \beta_1 \sum_{i=0}^{t-1} \gamma (1-\gamma)^i X_{t-i} - (1-\gamma)^t (Y_0 - \beta_0).$$

These equations, (9), are in terms of finite sums but still remain highly nonlinear in the unknown parameters. To solve these we need starting values of the parameters and a method of iteration. Probably the best method of iteration is the Newton method. Two other methods the same as the Newton method are the Newton-Raphson method and R.A. Fisher's method for solving maximum likelihood equations. Using any of these methods can be a lengthy and difficult process. Difficulties in calculation arise from such terms as

$$\sum_{i=0}^{t-1} \gamma (1-\gamma)^i X_{t-i} \text{ and } \sum_{i=1}^{t-1} i \gamma (1-\gamma)^{i-1} X_{t-i}.$$

What starting values of the parameters to use in starting the iteration process can also be a difficult problem.

The considerations above show that the least squares estimates are difficult to obtain. An alternative but more convenient method of estimation will not now be presented.

In many economic situations in which the model might be presumed to hold, e.g., absence of price supports, the prices X_t , may be smoothed fairly successfully with a low order polynomial, i.e.,

- (1) $X_t = a_0 + a_1 t$
 - (2) $X_t = a_0 + a_1 t + a_2 t^2$
 - (3) $X_t = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
- etc.

Let us take (1) and see the consequences.

Let:

$$X_t = a_0 + a_1 t$$

then

$$\begin{aligned} X^*_0 &= \sum_{i=0}^{\infty} \gamma (1-\gamma)^i X_{-i} = \sum_{i=0}^{\infty} \gamma (1-\gamma)^i (a_0 - a_1 i) \\ &= a_0 \sum_{i=0}^{\infty} \gamma (1-\gamma)^i - a_1 \sum_{i=0}^{\infty} i \gamma (1-\gamma)^i. \end{aligned}$$

Since

$$\sum_{i=0}^{\infty} \gamma (1-\gamma)^i = \sum_{i=0}^{\infty} i \gamma (1-\gamma)^i = \frac{1-\gamma}{\gamma},$$

$$X^*_0 = a_0 - a_1 \left(\frac{1-\gamma}{\gamma} \right),$$

then

$$\begin{aligned} X^*_t &= \sum_{i=0}^{t-1} \gamma (1-\gamma)^i X_{t-i} + (1-\gamma)^t X^*_0 \\ &= \sum_{i=0}^{t-1} \gamma (1-\gamma)^i [a_0 + a_1(t-i)] + (1-\gamma)^t X^*_0. \end{aligned}$$

Also, since

$$\sum_{i=0}^{t-1} \gamma(1-\gamma)^i = 1 - (1-\gamma)^t$$

and

$$\begin{aligned} \sum_{i=0}^{t-1} i \gamma(1-\gamma)^i &= \sum_{i=0}^{\infty} i \gamma(1-\gamma)^i - \sum_{i=0}^{\infty} (t+j) \gamma(1-\gamma)^{t+j} \\ &= \frac{1-\gamma}{\gamma} - (1-\gamma)^t \left[t \sum_{j=0}^{\infty} \gamma(1-\gamma)^j + \sum_{j=0}^{\infty} j \gamma(1-\gamma)^j \right] \\ &= \frac{1-\gamma}{\gamma} - (1-\gamma)^t \left[t + \frac{1-\gamma}{\gamma} \right], \end{aligned}$$

then

$$\begin{aligned} X_t^* &= a_0 [1 - (1-\gamma)^t] + a_1 t [1 - (1-\gamma)^t] - \\ &\quad a_1 \left[\frac{1-\gamma}{\gamma} - (1-\gamma)^t \left(t + \frac{1-\gamma}{\gamma} \right) \right] \\ &\quad + (1-\gamma)^t \left[a_0 - a_1 \frac{(1-\gamma)}{\gamma} \right], \\ &= a_0 - a_1 \frac{(1-\gamma)}{\gamma} + a_1 t. \end{aligned} \tag{11}$$

Therefore,

$$Y_t = \beta_0 + \beta_1 a_0 - \frac{a_1 \beta_1 (1-\gamma)}{\gamma} + a_1 \beta_1 t + \epsilon_t. \tag{12}$$

Now let

$$X_t = a_0 + a_1 t + a_2 t^2,$$

then

$$\begin{aligned} X_0^* &= \sum_{i=0}^{\infty} \gamma(1-\gamma)^i X_{-i} = \sum_{i=0}^{\infty} \gamma(1-\gamma)^i (a_0 - a_1 i + a_2 i^2) \\ &= a_0 - a_1 \frac{(1-\gamma)}{\gamma} + a_2 \frac{(2-\gamma)(1-\gamma)}{\gamma^2}, \end{aligned}$$

and

$$\begin{aligned} X_t^* &= \sum_{i=0}^{t-1} \gamma(1-\gamma)^i X_{t-i} + (1-\gamma)^t X_0^* \\ &= \sum_{i=0}^{t-1} \gamma(1-\gamma)^i [a_0 + a_1(t-i) + a_2(t-i)^2] + (1-\gamma)^t X_0^* \\ &= a_0 - a_1 \frac{(1-\gamma)}{\gamma} + a_1 t + \frac{(1-\gamma)^t a_2 (2-\gamma)(1-\gamma)}{\gamma^2} \\ &\quad + a_2 \sum_{i=0}^{t-1} \gamma(1-\gamma)^i (t-i)^2 \end{aligned} \tag{31}.$$

by using the results of the linear case.

Now,

$$\begin{aligned} &\sum_{i=0}^{t-1} \gamma(1-\gamma)^i (t-i)^2 - t^2 \sum_{i=0}^{t-1} \gamma(1-\gamma)^i - 2t \sum_{i=0}^{t-1} i\gamma(1-\gamma)^i - \sum_{i=0}^{t-1} i^2 \gamma(1-\gamma)^i \\ &= t^2 [1 - (1-\gamma)^t] - 2t \left[\frac{1-\gamma}{\gamma} - (1-\gamma)^t \left[t + \frac{1-\gamma}{\gamma} \right] \right] \\ &\quad + \sum_{i=0}^{t-1} i^2 \gamma(1-\gamma)^i \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{t-1} i^2 \gamma(1-\gamma)^i &= \sum_{i=0}^{\infty} i^2 \gamma(1-\gamma)^i - \sum_{j=0}^{\infty} (t+j)^2 \gamma(1-\gamma)^{t+j} \\ &= \frac{(2-\gamma)(1-\gamma)}{\gamma^2} - t^2 \sum_{j=0}^{\infty} \gamma(1-\gamma)^{t+j} \end{aligned}$$

$$\begin{aligned}
& - 2t \sum_{j=0}^{\infty} j \gamma (1-\gamma)^{t+j} - \sum_{j=0}^{\infty} j^2 \gamma (1-\gamma)^{t+j} \\
& = \frac{(2-\gamma)(1-\gamma)}{\gamma^2} - t^2 (1-\gamma)^t - 2t \frac{(1-\gamma)^{t+1}}{\gamma} \\
& - \frac{(1-\gamma)^{t+1}(2-\gamma)}{\gamma^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=0}^{t-1} \gamma (1-\gamma)^i (t-i)^2 & = t^2 \left[1 - (1-\gamma)^t \right] \\
& - 2t \left[\frac{1-\gamma}{\gamma} - (1-\gamma)^t \left(t + \frac{1-\gamma}{\gamma} \right) \right] \\
& + \frac{(2-\gamma)(1-\gamma)}{\gamma^2} - t^2 (1-\gamma)^t - \frac{2t(1-\gamma)^{t+1}}{\gamma} - \frac{(1-\gamma)^{t+1}(2-\gamma)}{\gamma^2} \\
& = \frac{(2-\gamma)(1-\gamma)}{\gamma^2} \left[1 - (1-\gamma)^t - 2t \frac{(1-\gamma)}{\gamma} - \frac{(1-\gamma)^{t+1}}{\gamma} + \frac{(1-\gamma)^{t+1}}{\gamma} \right] \\
& + t^2 \left[1 - (1-\gamma)^t + 2(1-\gamma)^t - (1-\gamma)^t \right]
\end{aligned}$$

which reduces to

$$\sum_{i=0}^{t-1} \gamma (1-\gamma)^i (t-i)^2 = \frac{(2-\gamma)(1-\gamma)}{\gamma^2} \left[1 - (1-\gamma)^t \right] - \frac{2(1-\gamma)}{\gamma} t + t^2.$$

Therefore

$$\begin{aligned}
X_t^{\circ} = a_0 - \frac{a_1(1-\gamma)}{\gamma} + \frac{a_2(2-\gamma)(1-\gamma)}{\gamma^2} \\
+ \left[a_1 - \frac{2(1-\gamma)}{\gamma} a_2 \right] t + a_2 t^2
\end{aligned}$$

and

$$Y_t = \beta_0 + \beta_1 \left[a_0 - \frac{a_1 (1-\gamma)}{\gamma} + \frac{a_2 (2-\gamma) (1-\gamma)}{\gamma^2} \right] \\ + \beta_1 \left[a_1 - \frac{2 (1-\gamma)}{\gamma} a_2 \right] t + a_2 \beta_1 t^2 + \epsilon_t \quad (14)$$

or

$$Y_t = \xi_0 + \xi_1 t + \xi_2 t^2 + \epsilon_t, \text{ say.} \quad (15)$$

The ξ_i in equation (15) can be estimated by least squares and setting them equal to the corresponding terms in equation (14) we have three equations for the three unknowns β_0 , β_1 , and γ . The a_i are assumed known, i.e., that estimates of the a_i have been obtained by least squares from equations (10).

4. **Conclusions.** The forms of equations (12) and (4) suggest that the expressions for higher order smoothed price functions will be easy to write down directly e.g., the constant term for the n th order smoothed price function can be obtained as

$$\xi_0 = \beta_0 + \beta_1 \sum_{i=0}^n a_i (-1)^i \mu'_i,$$

where

μ'_i is the i th moment about the origin of the geometric distribution, and

$$\xi_1 = \beta_1 \sum_{i=1}^n (-1)^{i-1} (i) (\mu'_{i-1}) a_i, \text{ etc.}$$

The estimation should proceed by a two-stage operation, i.e., first smooth the price data with a low order polynomial, then use these values to estimate β_0 , β_1 , and γ as above. The linear smoothing of course will not enable one to estimate β_0 , β_1 , and γ . We need at least a quadratic. Higher order poly-

nomials would allow some coefficients to be reestimated using the Y_t values. For example, a cubic fit on prices could allow only the a_0 , a_1 , and a_2 to be assumed and the a_3 to be reestimated directly from the final Y_t function.

If we use, say, the quadratic price function, simultaneous confidence limits can be constructed for β_0 , β_1 , and γ under the obvious assumptions needed about a_0 , a_1 , and a_2 ; i.e., fitting equation (15), we put simultaneous limits on ξ_0 , ξ_1 , and ξ_2 and since these functions of β_0 , β_1 , and γ , we obtain simultaneous limits from the substitution and inversion of these limits, conditional on observed values of a_0 , a_1 , and a_2 .

Finally, if desired, the values β_0 , β_1 , and γ obtained from, say a quadratic fit on prices could be used subsequently in an iterative scheme, e. g., the Newton method, on the original "unsmoothed" function as starting values.

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REFERENCES

- FISHER, I. "Our stable dollar and the so-called business cycle", *J. American Statistical Association*, 21 (1925) p. 179.
- FISHER, R. A. "On the mathematical foundations of theoretical statistics", *Philosophical Transactions of the Royal Society*, London, Ser. A, 222 (1922) pp. 309-368.
- KLEIN, L. R. "The estimation of distributed lags", *Econometrica*, 26 (1958) pp. 553-565.
- KOYCK, L. M. *Distributed Lags and Investment Analysis*. Amsterdam: North Holland Publishing Company, 1954.
- NERLOVE, M. *Distributed Lags and Demand Analysis*. Washington: U.S. Dept. Agr. Handbook No. 141, U.S. Government Printing Office, 1958.
- SCARBOROUGH, J. B. *Numerical Mathematical Analysis*. 4th Edition. Baltimore: The Johns Hopkins Press, 1960.